FIXED POINT THEOREMS ON INFINITE DIMENSIONAL MANIFOLDS

BY

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Introduction. Let X be a differentiable manifold of finite or infinite dimension, f a continuous mapping of X into X. f is said to be a compact mapping if $f(X)$ is relatively compact in X while f is said to be locally compact if each point x of X has a neighborhood N_x such that $f(N_x)$ is relatively compact in X.

We are concerned in the present paper with sufficient conditions for the existence of fixed points of compact and locally compact mappings of manifolds, especially those of infinite dimension. The questions which we wish to answer are the following:

(1) *If f is a compact self-mapping of an acyclic manifold X, does f have a fixed point in X?*

(2) *If f is a compact self-mapping of a manifold X, does the Lefschetz fixed point theorem hold for f?*

(3) *Let f be a locally compact mapping of a manifold X such that for some positive integer m, $f^m(X)$ is relatively compact in X. Then can we assert that f has a fixed point?*

For Banach spaces X, the affirmative answer to question (1) is the well-known Schauder fixed point theorem [21]. An affirmative result for question (3) for Banach spaces was obtained by the writer in [3]. For Banach spaces, question (2) is obliquely related to the Leray-Schauder theory of topological degree [20] for completely continuous displacements.

The principal results of the present paper are affirmative answers to questions (1), (2), and (3) for a broad class of manifolds which includes all manifolds which can be imbedded as neighborhood retracts in Banach spaces. More specifically we have the following:

THEOREM (THEOREM 3 OF §1). *Let f be a compact mapping of the metric space X where X can be imbedded as a neighborhood retract of a Banach space E. Then the Lefschetz number*

$$\Lambda(f) = \sum_{s=0}^{\infty} (-1)^s \operatorname{tr}(f_{*s})$$

is well-defined (where f_{*s} is the homomorphism induced by f on $H_s(X)$, the s-dimensional singular homology group of X with rational coefficients). If $\Lambda(f) \neq 0$, f has a fixed point in X. If X is acyclic, $\Lambda(f) = 1$.

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Some examples of acyclic manifolds X satisfying the hypotheses of the above theorem are the following: Let E be an infinite dimensional Banach space, $S(E)$ the unit sphere $\{u \mid u \in E, \|u\| = 1\}$ in E, $D(E)$ the open (or closed) unit ball in E. Then

$$X_1 = E - \bigcup_\gamma Y_\gamma,$$

$$X_2 = S(E) - \bigcup_\gamma Y_\gamma,$$

$$X_3 = D(E) - \bigcup_\gamma Y_\gamma,$$

$$X_4 = U(H) - \bigcup_\gamma Y_\gamma$$

are examples, where H is an infinite dimensional Hilbert space, $U(H)$ the unitary group on H in the norm topology, and in each space, $\{Y_\gamma\}$ is a locally finite family of open subsets homeomorphic by h_γ to open balls in infinite dimensional Banach spaces E_γ such that h_γ can be extended to a homeomorphism of the boundaries.

As far as question (3) is concerned we have the following result:

THEOREM (THEOREM 11 OF §2). *Let X be a space which can be imbedded as a retract of a space Y such that for each compact subset K of Y there exist a compact acyclic set A_1 and a compact absolute neighborhood retract A_2 both containing K. (In particular, Y may be any Banach space.) Let f be a locally compact self-mapping of X with $f^m(X)$ relatively compact in X for some positive integer m. Then f has a fixed point in X.*

X_1, X_2, and X_3 above are examples of spaces satisfying the hypotheses of this last theorem.

§1 is devoted to the study of compact mappings. We give a general theory of the Lefschetz number for compact mappings of a topological space X which can be imbedded as a retract of a space X_1 such that every compact subset K of X is contained in a Lefschetz space Y contained in X_1. (Y is a Lefschetz space if the Lefschetz fixed point theorem holds for all continuous self-mappings of Y.) In Theorem 1, we show that the Lefschetz fixed point theorem holds for all compact mappings of such spaces. In a Corollary, we observe that if such a space is acyclic, every compact self-mapping f of X has a fixed point. The other results of §1, follow by specialization from Theorem 1. In Theorems 8 and 9, we show that every closed C^3 submanifold of a Banach space E admitting C^3 partitions of unity falls in this class of spaces(2).

(2) *Added in proof.* The writer has been informed in a recent letter from J. Ellis (commenting upon a preprint of the present paper) that the latter had earlier obtained independently a proof of a result stronger than Theorems 8 and 9, namely for C^0 manifolds modeled on F-spaces. This result follows directly from Theorem 3 if one uses the known result (which may be derived easily from results of Hanner) that such manifolds may be imbedded as neighborhood retracts of Banach spaces.

§2 is devoted to the study of locally compact mappings such that $f^m(X)$ is relatively compact for some $m > 1$. Here the basic general result is Theorem 11 stated above.

The extensive use of retraction properties in the present paper is related in its general concept to its slightly different use in the study of the local fixed point index by Browder [2], Leray [18], [19], and Deleanu [8]. The proofs of the results of §2 are an adaptation and extension of the techniques applied in Banach spaces by the writer in Browder [2], [6], [7]. Other results on the local fixed point index are developed in Leray [17] and Browder [4], [5]. We apply general properties of compact metric absolute neighborhood retracts as given in Kuratowski [13] and Lefschetz [16] and, in particular, our discussion is based on the classical forms of the Lefschetz fixed point theorem for such spaces as established in Lefschetz [15], [16]. In applying our results to particular infinitedimensional manifolds, we make essential use of theorems of Dugundji [9] and of Kuiper [12].

A recent survey of results in general fixed point theory is given by Van der Walt [22].

The writer is indebted to Stephen Smale for stimulating the writer's interest in fixed point theorems for infinite dimensional manifolds and for making him acquainted with some of the results of [1] and [14] applied at the conclusion of §1.

1. Let X be a topological space. For each integer $s \geqq 0$, we denote by $H_s(X)$ the s-dimensional singular homology group of X with coefficients in the field F of rational numbers. If f is a continuous mapping of X into Y, we denote by f_{*s} the induced homomorphism of $H_s(X)$ into $H_s(Y)$. $H_s(X)$ and $H_s(Y)$ are vector spaces over F and f_{*s} is a linear mapping of $H_s(X)$ into $H_s(Y)$.

DEFINITION 1. *If V is a vector space over F (of possibly infinite dimension), h a linear mapping of V into V, then h is said to have a trace if $h(V)$ is of finite dimension. The trace of h, $\mathrm{tr}_V(h)$, is defined to be equal to the trace of h restricted to any subspace V_1 of finite dimension in V which contains $h(V)$.*

To deal with the properties of the trace, we need the following well-known elementary fact:

LEMMA 1. *Let V_1 and V_2 be vector spaces of finite dimension, h_1 a linear map of V_1 into V_2, h_2 a linear map of V_2 into V_1.*
Then

$$\mathrm{tr}_{V_1}(h_2 \circ h_1) = \mathrm{tr}_{V_2}(h_1 \circ h_2).$$

DEFINITION 2. *Let f be a continuous self-mapping of X. Then the Lefschetz number $\Lambda(f)$ of f is said to exist if $f_{*s} = 0$ except for a finite number of $s \geqq 0$, while f_{*s} has a trace in the sense of Definition 1 for each $s \geqq 0$.*

The Lefschetz number $\Lambda(f)$ is then defined by

$$\Lambda(f) = \sum_{s=0}^{\infty} (-1)^s \operatorname{tr}(f_{*s}) \quad .$$

DEFINITION 3. *Y is said to be a Lefschetz space if for each continuous self-mapping f of Y, $\Lambda(f)$ is defined and if $\Lambda(f) \neq 0$, then f has a fixed point.*

THEOREM 1. *Let f be a continuous mapping of the topological space X into X. Suppose that there exists a topological space X_1, a subset Y of X_1, and an imbedding j of X in X_1 such that all the following conditions hold:*

(a) *$j(X)$ is a retract of X_1,*

(b) *Y is a Lefschetz space,*

(c) *$j(f(X)) \subset Y$.*

Then $\Lambda(f)$ exists and if $\Lambda(f) \neq 0$, f has a fixed point in X.

COROLLARY TO THEOREM 1. *Suppose that in addition to the hypotheses of Theorem 1, we assume also that X is pathwise connected and one of the following conditions hold:*

(i) *every singular s-cycle in $f(X)$ for $s \geqq 1$ bounds in X,*

(ii) *$f(X)$ is contractible to a point in X.*

Then $\Lambda(f) = 1$ and f has a fixed point.

Before proceeding to the proof of Theorem 1, we remark the following useful fact:

LEMMA 2. *If Y is a Lefschetz space, then $H_s(Y)$ is of finite dimension for all $s \geqq 0$ and $H_s(Y) \neq 0$ only for a finite number of values of $s \geqq 0$.*

Proof of Lemma 2. If ι_Y is the identity map of Y, then $\Lambda(\iota_Y)$ exists. Q.E.D.

Proof of Theorem 1. By identifying X with its image under j in X_1, we may assume without loss of generality that X is a subset of X_1, j is the injection map of X into X_1, and f maps X into a subset of $X \cap Y$.

Let k be the injection map of Y into X_1, r the retraction map of X_1 on X. By considering the image of x in X under f as an element of Y we obtain from f a map f_1 of X into Y. Set

$$\begin{aligned} g &= j \circ f \circ r \colon X_1 \to X_1, \\ g_1 &= f_1 \circ r \colon X_1 \to Y, \\ g_2 &= g_1 \circ K \colon Y \to Y. \end{aligned}$$

Then

$$g_{*s} = j_{*s} \circ (f \circ r)_{*s}$$

and since Y is a Lefschetz space, the image of j_{*s} is of finite dimension and $\neq 0$ only for a finite number of values of s, $s \geqq 0$. Hence the same is true for g_{*s}, i.e., $\Lambda(g)$ is well-defined.

Since $f = r \circ g \circ j$, we have for all $s \geqq 0$,

$$f_{*s} = r_{*s} \circ g_{*s} \circ j_{*s}.$$

Since g_{*s} has a finite dimensional image for each $s \geqq 0$, and $g_{*s} = 0$ except for a finite number of values of s, the same is true of f_{*s}. Hence $\Lambda(f)$ exists.

Since $g = j \circ f \circ r$, we have

$$g_{*s} = j_{*s} \circ f_{*s} \circ r_{*s}.$$

From the fact that $r \circ j = \iota_X$, we have

$$r_{*s} \circ j_{*s} = \iota_{*s}$$

so that j_{*s} is injective and $H_s(X_1)$ splits into the sum

$$H_s(X_1) = j_{*s}(H_s(X)) + \operatorname{Ker}(r_{*s}).$$

On $j_{*s}(H_s(X))$, g_{*s} may be written as

$$g_{*s} = j_{*s} f_{*s} (j_{*s})^{-1}.$$

Hence $\operatorname{tr}(g_{*s}) = \operatorname{tr}(f_{*s})$ for each $s \geqq 0$, so that $\Lambda(g) = \Lambda(f)$.

Suppose now that $\Lambda(f) \neq 0$. Then $\Lambda(g) \neq 0$. On the other hand,

$$g = k \circ g_1$$

while

$$g_2 = g_1 \circ k.$$

Hence

$$g_{*s} = k_{*s} \circ (g_1)_{*s}$$

and

$$(g_2)_{*s} = (g_1)_{*s} \circ k_{*s}.$$

By the definition of the trace (Definition 1),

$$\operatorname{tr}(g_{*s}) = \operatorname{tr}_{V_s}(g_{*s})$$

where $V_s = k_{*s}(H_s(Y))$. V_s is of finite dimension since Y is a Lefschetz space. We have

$$(g_1)_{*s} \colon V_s \to H_s(Y),$$

$$h_{*s} \colon H_s(Y) \to V_s$$

and

$$(g_{*s} \mid V_s) = k_s \circ (g_1)_{*s},$$

$$(g_2)_{*} = (g_1)_{*s} \circ k_s.$$

Applying Lemma 1 with $V_1 = V_s$, $V_2 = H_s(Y)$, we find that

$$\mathrm{tr}((g_2)_{*s}) = \mathrm{tr}(g_{*s} \mid V_s) = \mathrm{tr}(g_{*s})$$

so that

$$\Lambda(g_2) = \Lambda(g).$$

If $\Lambda(g) \neq 0$, $\Lambda(g_2) \neq 0$. Since g_2 is a continuous self-mapping of the Lefschetz space Y, g_2 has a fixed point y_0 in Y. Hence $y_0 = f(r(y_0))$ so that

$$y_0 = f(y_0). \qquad \text{Q.E.D.}$$

Proof of the Corollary to Theorem 1. The mapping f may be written in the form

$$f = j_1 \circ f_1,$$

where f_1 is the obvious map of X into $f(X)$ induced by f, j_1 the injection map of $f(X)$ into X.

If (i) holds, $j_{*s} = 0$ for every $s \geqq 1$. Hence $f_{*s} = 0$ for every $s \geqq 1$. Thus

$$\Lambda(f) = \mathrm{tr}(f_{*0}) = 1,$$

and the existence of a fixed point for f follows from Theorem 1. Obviously condition (ii) implies condition (i). Q.E.D.

LEMMA 3. *The following types of topological spaces are all Lefschetz spaces*:

(a) *compact metric absolute neighborhood retracts*;

(b) *convexoid spaces* (*i.e., compact spaces having arbitrarily fine finite closed coverings each of whose elements is acyclic and such that any nonempty intersection of elements of a given covering is also an element of that covering*);

(c) *a finite union of compact convex subsets of a Banach space*;

(d) *a finite union of compact convex subsets of a locally convex linear space E*;

(e) *retracts of Lefschetz spaces.*

Proof of Lemma 3. Part (a) was proved by Lefschetz [16]. Part (b) is included in the results of Leray [17].

For the proof in case (c)(3) it suffices, by the result already obtained for case (a), to show that each finite union of compact convex subsets of a Banach space E is a compact metric absolute neighborhood retract (ANR). We note first that by a result of Klee [11], every compact convex subset of a Banach space is homeomorphic to the Hilbert parallelotope and hence is an absolute retract. By a theorem of Aronszajn and Borsuk (cf. [13, p. 260], if A_1 and A_2 are two compact ANR's contained in a space E and if $A_1 \cap A_2$ is an ANR, then so is $A_1 \cup A_1$. If $\{K_1, \cdots, K_m\}$ is our family of compact convex subsets of E, we may assume without loss of generality that each nonempty intersection of a subfamily of

(3) Strictly speaking, the result of (d) includes (c) but we give a simpler and more easily verifiable proof for this case which is the essential one in the applications below.

the K_j's is also a member of the family and that if $K_j \subset K_k$, then $j \leqq k$. We then prove by induction on r ($r \leqq m$) that each union of the form

$$\bigcup_k K_{j(k)}; \quad j(k) \leqq r$$

is an ANR. For $r = 1$, the result follows from the fact that K_1 is an absolute retract. Suppose it is true for $(r-1)$. To show that it is true for r, we must show that

$$\left(\bigcup_{j(k) < r-1} K_{j(k)} \right) \cup K_r$$

is an ANR. The union in the round bracket is an ANR by the inductive assumption. We have

$$K_r \cap \left(\bigcup_{j(k) \leqq r-1} K_{j(k)} \right) = \bigcup_{j'(k) \leqq \ -1} K_{j(k)},$$

which is also an ANR by the inductive assumption. Hence by the theorem of Aronszajn and Borsuk, $\bigcup_{j(k) \leqq r} K_{j(k)}$ is also an ANR, the inductive step is proved, and each finite union of compact convex subsets of a Banach space is an ANR.

For the proof in case (d), we observe that each compact convex subset K in a locally convex linear topological space is contractible. Moreover, given an open covering α of K, we may find a finite covering β of K by closed convex sets which refines α. If $\beta = \{N_1, \cdots, N_m\}$ then we may replace N_j by $K_j = N_j \cap K$ to obtain a covering by compact convex sets, each of which is contractible and all of whose intersections are contractible. Since contractible sets are acyclic, K is a convexoid space. Moreover, any finite union of K's may be decomposed as above into a finite union of small compact convex sets. Hence each finite union of compact convex subsets of a locally convex linear space is convexoid. Since convexoid spaces are Lefschetz spaces by part (b), the conclusion of (d) follows.

To prove (e), let X be a Lefschetz space, Y a retract of X, r the retraction map of X on Y, j the injection map of Y into X. Since $r \circ j = \iota_Y$, we have $r_{*s} \circ j_{*s} = \iota_{*s}$ and since the image of j_{*s} is finite dimensional and nontrivial only for finitely many $s \geqq 0$, it follows that $H_s(Y)$ is of finite dimension and $H_s(Y) \neq 0$ only for finitely many values of s. Thus for a self-mapping f of Y, $\Lambda(f)$ is always defined.

Let $g = j \circ f \circ r : X \to X$. For each $s \geqq 0$,

$$g_{*s} = j_{*s} \circ f_{*s} \circ r_{*s}.$$

By Lemma 1,

$$\begin{aligned} \operatorname{tr}(g_{*s}) &= \operatorname{tr}(j_{*s} \circ (f_{*s} \circ r_{*s})) \\ &= \operatorname{tr}(f_* \circ r_* \circ j_*) = \operatorname{tr}(f_*). \end{aligned}$$

Hence

$$\Lambda(f) = \Lambda(g).$$

If $\Lambda(f) \neq 0$, we have $\Lambda(g) \neq 0$ so that g has a fixed point x_0 since X is a Lefschetz space. However

$$x_0 = f(r(x_0))$$

so that $x_0 \in Y$, $r(x_0) = x_0$, and x_0 is a fixed point of f. Q.E.D.

DEFINITION 3. *A continuous map f of X into X is said to be compact if $f(X)$ is contained in a compact subset of X.*

DEFINITION 4. *A compact mapping f of X into X is said to be smooth if there exists an imbedding j of X in a space X_1 such that $j(X)$ is a retract of X_1 and there exists a Lefschetz space Y contained in X_1 such that*

$$j(f(X)) \subset Y.$$

THEOREM 2. *Let f be a compact smooth mapping of X into X. Then $\Lambda(f)$ exists and if $\Lambda(f) \neq 0$, f has a fixed point in X.*

If in addition, every singular cycle in $f(X)$ bounds in X and X is pathwise connected, then $\wedge(f) = 1$ and f has a fixed point in X.

Proof of Theorem 2. This is just a restatement in the new terminology of Theorem 1 and its Corollary.

THEOREM 3. *Let f be a compact mapping of X into X and suppose that X can be imbedded in a Banach space E so that X is the retract of an open neighborhood U of X in E. Then f is smooth, $\Lambda(f)$ is well-defined, and if $\Lambda(f) \neq 0$, f has a fixed point in X. If X is acyclic $\Lambda(f) = 1$.*

Proof of Theorem 3. Let K be a compact subset of X containing $f(x)$. It suffices by Theorem 2 to construct a Lefschetz space Y contained in U which contains K. We construct Y as the union of a finite number of compact convex subsets of E, which is a Lefschetz space by Lemma 3(c). Each point x of K has a closed convex neighborhood N_x contained in U. A finite family of such neighborhoods $\{N_1, N_2, \cdots, N_m\}$ covers K. Let K_j be the convex closure of the compact set $K \cap N_j$. K_j is compact and convex, while $K_j \subset N_j \subset U$. Hence

$$Y = \bigcup_{j=1}^{m} K_j .$$

is our desired Lefschetz space. Q.E.D.

A similar argument using Lemma 3(d) instead of Lemma 3(c) gives the following result:

THEOREM 4. *Let f be a compact continuous self-mapping of the space X and suppose that X may be imbedded as a neighborhood retract in the locally* *convex linear topological space E, where E is quasi-complete (i.e., bounded closed sets of E are complete). Then f is smooth, $\Lambda(f)$ exists, and if $\Lambda(f) \neq 0$, f has a fixed point in X.*

THEOREM 5. *Let E be an infinite dimensional Banach space, $S(E)$ its unit sphere*

$$S(E) = \{u \mid u \in E, \ \|u\| = 1\}.$$

Let f be a compact mapping of $S(E)$ into itself. Then f has a fixed point.

Proof of Theorem 5. By a theorem of Dugundji [9], there exists a retraction of the closed disk $D(E) = \{u \mid u \in E, \ \|u\| \leqq 1\}$ onto $S(E)$. Since $D(E)$ is a neighborhood retract in E and is contractible, it follows by Theorem 3 that f is smooth, $\Lambda(f)$ exists, and $S(E)$ is acyclic.

Hence $\Lambda(f) = 1$ and f has a fixed point. Q.E.D.

THEOREM 6. *Let X be any one of the following:*

(a) *an infinite dimensional Banach space E.*

(b) $S(E)$,

(c) $D(E)$,

(d) *an acyclic neighborhood retract in a Banach space E.*

Suppose that $\{Y_\gamma\}$ is a locally finite family of open subsets of X which are pairwise disjoint and such that for each γ, $\bar{Y}_\gamma$ (the closure of Y in X) is homeomorphic to $D(E_\gamma)$ for some infinite dimensional Banach space E_γ under a homeomorphism h which carries $S(E_\gamma)$ onto the boundary $\dot{Y}$ of Y_γ in X.

Let $X_2 = X - \bigcup_\gamma Y_\gamma$ and let f be a compact map of X_2 into X_2. Then f has a fixed point in X_2.

Proof of Theorem 6. By Theorem 3, it suffices to show that X_2 is a retract of X. We define the retraction r by noting that by the theorem of Dugundji [9] applied above, $\dot{Y}_\gamma$ is a retract of $\bar{Y}_\gamma$. Q.E.D.

THEOREM 7. *Let H be an infinitely dimensional Hilbert space, $U(H)$ the group of unitary operators on H in the norm topology. Let f be a compact continuous map of $U(H)$ into itself. Then f has a fixed point.*

Proof of Theorem 7. Let $L(H)$ be the space of bounded linear operators on H with the norm topology, $GL(H)$ the open subset of $L(H)$ consisting of invertible operators, $U(H)$ the subset of unitary operators. To prove Theorem 6, it suffices by Theorem 2 and 3 to prove that $U(H)$ is a retract of $GL(H)$ and that for $s \geqq 1$, each singular cycle in $U(H)$ bounds. The second fact follows from a recent result of N. Kuiper [12] that $\pi_j(U(H)) = 0$ for all $j \geqq 1$ together with the Hurewicz theorem. For the first, we remark that every A in $GL(H)$ may be written uniquely in the form

$$A = U|A|$$

where $|A| = (A^*A)^{1/2}$. If the map $A \to U$ is continuous in norm, it provides the necessary retraction.

To prove that the mapping $A \to U$ is norm continuous, it suffices since $U = A|A|^{-1}$ to show that the map $A \to |A|$ is continuous in norm. Since $|A|^2 = A^*A$ where the latter is norm-continuous in A, it suffices to show that the map $|A|^2 \to |A|$ is norm-continuous or more generally that over the set of bounded self-adjoint operators B with $B \geqq I$, the mapping $B \to B^{1/2}$ is norm-continuous.

Let C be a smooth Jordan curve in the complex plane which contains the segment $[1, \|B_0\|]$ in the interior and is symmetric under the map $\zeta \to \bar{\zeta}$. For self-adjoint operators B close to B_0 in norm, the spectrum of B will lie inside C. We form

$$B_1 = \frac{1}{2\pi i} \int_C \lambda^{1/2}(\lambda I - B)^{-1} d\lambda.$$

Then $B_1^* = B_1$, $B_1^2 = B$ and $B_1 \geqq 0$. Hence B_1 is the unique positive square root of B. It follows easily from the integral formula that $B_1 = B^{1/2}$ is norm continuous in B. Q.E.D.

By an infinite dimensional manifold modeled on the Banach space E is meant (Lang [14]) a Hausdorff space X carrying an atlas of coordinate charts on B, i.e., with a given family $\{U_j, \phi_j\}$ where U_j is an open subset of X and ϕ_j is a homeomorphism of U_j on the open unit ball B in E ($B = \{u \mid u \in E, \|u\| < 1\}$). X is said to be a manifold of class C^s, $s \geqq 1$, if for each pair of charts (U_j, ϕ_j) and (U_k, ϕ_k) such that $U_j \cap U_k \neq \varnothing$, the mapping $\phi_k \phi_j^{-1}$ is a diffeomorphism of class C^s of $\phi_j(U_j \cap U_k)$ onto $\phi_k(U_j \cap U_k)$. If X and Y are manifolds of class C^s, the continuous mapping f of X into Y is said to be of class C^s if for each coordinate chart (U_j, ϕ_j) on X and (V_k, ψ_k) on Y such that $f(U_j) \cap V_k \neq \varnothing$, the mapping

$$\psi_k \circ f \circ \phi_j^{-1}$$

is a mapping of class C^s from $\phi_j(U_j \cap f^{-1}(V_k))$ into $\psi_k(V_k)$.

If X is a manifold of class C^s, $s \geqq 1$, x a point of X, we consider curves of class C starting at x (i.e., C^s-maps g of the unit interval $[0,1]$ into X such that $g(0) = x$). Two such curves g and g_1 are said to be equivalent if for a coordinate chart (U, ϕ) at x, $\phi \circ g$ and $\phi \circ g_1$ have the same tangent vector at 0. The tangent space T_x to X at x consists of the set of equivalence classes of curves in X at x with a linear structure obtained from its identification with the tangent vectors of the curves $\phi \circ g$ at $\phi(x)$ in E. To each differentiable map f of X into Y, there corresponds a linear map df_x of T_x, the tangent space to X at x, into $T_{f(x)}$, the tangent space to Y at $f(x)$, where df_x assigns to the equivalence class of curve $g: (I,0)\to(X,x)$ in X, the equivalence class of the curve $f\circ g: (I,0)\to(Y,f(x))$.

A mapping h of X into Y of class C^s, $s \geqq 1$, is said to be an imbedding of X as a submanifold of class C^s in Y if h is homeomorphism of X on $h(X)$ and if for each point x of X, df_x is an injective map of T_x into $T_{f(x)}$. We say that X is imbedded as a closed submanifold of Y if $h(X)$ is closed in Y. An open neighborhood U of X in Y is said to be a tubular neighborhood if there exists a fibre bundle M over X whose fibre is a Banach space and an open neighborhood V of the zero section of M which is homeomorphic to U. We may obviously assume that X is a deformation retract of U.

Theorem 8. *Let X be a closed C^3-submanifold of the Banach space E, where E admits C^2 partitions of unity. Then X is a neighborhood retract of E and for every compact mapping f of X into X, $\Lambda(f)$ exists and the Lefschetz fixed point theorem is valid for f.*

Proof of Theorem 8. The retraction property follows from the theorem on the existence of a tubular neighborhood of X in E under the hypotheses of Theorem 8 [14], [1, p. 73]. The remaining conclusion follows from Theorem 3. Q.E.D.

Theorem 9. *Every paracompact separable C^3-manifold modeled on a separable Banach space E_1 with C^3 partitions of unity can be C^3-imbedded as a closed submanifold of a Banach space E with C^3 partitions of unity. For such a manifold, the conclusion of Theorem 8 holds.*

Proof of Theorem 9. The proof is essentially identical with the proof of Theorem 7.4 of [1] for the case where E_1 is a separable Hilbert space.

2. In the present section, we establish the extension of some of the results of §1 to mappings f which need no longer be compact but such that f is locally compact and f^m is compact for some integer $m > 1$. The results obtained below generalize and strengthen corresponding results of the writer [3], [6], [7] for self-mappings of Banach spaces.

Our basic tool is the following theorem, which was stated without proof as Theorem 3 of [7].

Theorem 10. *Let X_0 be a compact absolute neighborhood retract, X_1 an open subset of X_0, X_2 a compact subset of X_1. Let f be a continuous mapping of X_1 into X_0, m a positive integer such that f^m is defined on X_2 and*

$$f^m\left(\bigcup_{j=0}^{m-1} f^j(X_2)\right) \subset X_3 \subset \mathrm{Int}(X_2)$$

(where $\mathrm{Int}(X_2)$ is the interior of X_2 with respect to X_0) and X_3 is acyclic.

Then f has a fixed point in X_2.

Proof of Theorem 10. Let Ω be the directed set of finite open coverings α of X_0. For each α in Ω, let N_α denote the nerve of the covering α, so that each vertex v of the finite polytope N_α corresponds to an element V_v of α. For each α, we choose a canonical mapping p_α of X_0 into N_α.

For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha \in \Omega$ with $\mathrm{mesh}(\alpha) = \sup_{V \in \alpha} \mathrm{diam}(V) < \delta$, then there exists a continuous map q_α of N_α into X_0 such that the mapping $q_\alpha \circ p_\alpha$ moves each point of X_0 at most ε and is ε-homotopic to ι_{X_0} the identity map of X_0. For a given $\varepsilon > 0$, we choose such a mapping q_α for each α with mesh $< \delta$, where ε will be specified at a later point of the proof.

Let N'_α be the closed subcomplex of N_α of all simplexes of N_α all of whose vertices correspond to elements V of α whose stars in α are contained in $X_1 - M_\varepsilon(X_0 - X_1)$ where $M_\varepsilon(S)$ is the ε-neighborhood of S in X_0. Then q_α maps N'_α into X_1 and we may form the mapping

$$f_\alpha : N'_\alpha \to N_\alpha$$

given by

$$f_\alpha = p_\alpha \circ f \circ (q_\alpha \,|\, N'_\alpha).$$

Let N''_α be the closed subcomplex of N_α of all simplexes all of whose vertices correspond to elements V of α which intersect $M_\varepsilon(X_2)$.

We verify the following facts about f_α, N'_α, and N''_α if $\varepsilon > 0$ is sufficiently small and $\alpha \in \Omega$ is sufficiently fine:

(1) f^m_α is defined on N''_α, $f^m_\alpha(N''_\alpha) \subset N''_\alpha$ and

$$f^m_\alpha\left(\bigcup_{j=0}^{m-1} (f_\alpha)^j(N''_\alpha)\right) \subset \mathrm{Int}(N''_\alpha)$$

(where $\mathrm{Int}(N''_\alpha)$ is the interior of N''_α with respect to N''_α);

(2) N''_α lies in the interior of N'_α;

(3) f^m_α mapping $\bigcup_{j=0}^{m-1}(f_\alpha)^j(N''_\alpha)$ into N''_α is homologically trivial since it is the composition of a homologically trivial map h_α into X_2 with the map p_α of X_2 into N''_α. Here we may take $h_\alpha = f \circ q_\alpha f^{m-1}_\alpha \cong f^m \circ q_\alpha$.

Using (1), (2), and (3), we have essentially reduced the proof of Theorem 10 to the case when X_0, X_1, and X_2 are finite polytopes. We now perform a further reduction by replacing f_α by another mapping g_α of N'_α into N_α where g_α is a simplicial mapping of one barycentric subdivision $B_r(N'_\alpha)$ into another $B_s(N_\alpha)$ such that

(1′) g^m_α is defined on N''_α, $g^m_\alpha(N''_\alpha) \subset N''_\alpha$;

(2′) $g^m_\alpha(\bigcup_{j=0}^{m-1}(g_\alpha)^j(N''_\alpha)) \subset \mathrm{Int}(N''_\alpha)$;

(3′) g^m_α mapping $\bigcup_{j=0}^{m-1} (g'_\alpha)^j(N''_\alpha)$ into N''_α is homologically trivial;

(4) g^j_α, $0 \leqq j \leqq m-1$, is a simplicial map from the barycentric subdivision of N_α to another.

We set

$$A_\alpha = \bigcup_{j=0}^{m-1} (g_\alpha)^j(N''_\alpha).$$

By property (4), A_α is a finite union of simplexes and hence an absolute neighborhood retract. By property (1′), g_α maps A_α into A_α. By property (3′), g_α^m is a homologically trivial map of A_α into A_α.

LEMMA (2.1) (LEMMA 1 OF [3]). *If A is a compact absolute neighborhood retract, f a continuous self-mapping of A such that f^m is homologically trivial, then f has a fixed point on A.*

Proof of Lemma (2.1). Since $0 = (f^m)_{*s} = (f_{*s})^m$ for $s \geqq 1$, we know that $\operatorname{tr}(f_{*s}) = 0$ for $s \geqq 1$. Since $(f^m)_{*s}$ is the identity on one generator of $H_0(A)$, zero on the rest, $\operatorname{tr}(f_0) = 1$. Hence $\Lambda(f) = 1$ and f has a fixed point by the Lefschetz fixed point theorem.

Proof of Theorem 10 completed. By Lemma (2.1), g_α has a fixed point in A_α which must lie in N''_α. If w_α is such a fixed point, w and $f_\alpha(w_\alpha)$ must lie in the same simplex of N_α. Hence if $x_\alpha = g_\alpha(w_\alpha)$,

$$\operatorname{dist}(x_\alpha, g_\alpha f_\alpha(w_\alpha)) = (g_\alpha \circ p_\alpha)(f(x_\alpha))$$

is at most at distance ε from $f(x_\alpha)$. Hence if $3\operatorname{mes}(\alpha) < \varepsilon$, we have

$$\operatorname{dist}(x_\alpha, f(x_\alpha)) < 2\varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, there must exist a fixed point x_0 of f on X_2 (since all the fixed points x_α constructed above lie in the 3ε-neighborhood of X_2 in X_0). Q.E.D.

DEFINITION (2.1). *Let X be a topological space, f a continuous self-mapping of X. Then f is said to be locally compact if each point x of X has a neighborhood N_x in X such that $f(N_x)$ is relatively compact in X.*

THEOREM 11. *Let X be a topological space, f a locally compact self-mapping of X. Suppose that X has an imbedding j in a space Y such that $j(X)$ is a retract of Y which for each compact subset K of Y there exist sets A_1 and A_2 in Y containing K where A_1 is compact and acyclic while A_2 is a compact absolute neighborhood retract. Suppose that for some integer $m \geqq 1$, $f^m(X)$ is relatively compact in X.*

Then f has a fixed point in X.

Proof of Theorem 11. We may assume without loss of generality that X is a subset of Y and j is the injection map of X into Y. We may also assume without loss of generality that $X = Y$ since if r is the retraction map of Y onto X, then the self-mapping $g = j \circ f \circ r$ of Y satisfies all the hypotheses imposed on f and has the same fixed points as f.

Since $f^m(X)$ is relatively compact in X, its closure K_0 in X is compact. So is

$$K = \bigcup_{j=0}^{m-1} f^j(K_0)$$

and we see that $f(K) \subset K$.

Since we are assuming that $X = Y$, it follows from the hypothesis of Theorem 11 that there exists a compact acyclic set A_1 in X which contains K. Let

$$A_1' = \bigcup_{j=0}^{m-1} f^j(A_1).$$

Since A_1' is compact and f is a locally compact mapping, there exists a neighborhood U of A_1' in X such that $f(U)$ is contained in a compact subset K' of X. By the hypothesis of Theorem 11, there exists a compact absolute neighborhood retract A_2 in X which contains $A_1' \cup K'$.

We now apply Theorem 10 taking $X_0 = X_2$, $X_1 = A_2 \cap f^{-1}(A_2)$ and considering the mapping f_1 of X_1 into X_0 obtained by restricting f to $f^{-1}(A_2) \cap A_2$ and taking the restricted mapping as a mapping into A_2. We set $X_3 = A_1$. Since $f^m(X) \subset A_1$, it will certainly follow that f_1^m will map the domain of f_1^m into A_1. We note that A_1 lies in the domain of f_1^m since f^j maps A_1 into A_2 for all $j \leqq m$. Furthermore, the domain of f_1 contains A_1 in its interior since

$$A_1 \subset U \cap A_2 \subset f^{-1}(A_2) \cap A_2$$

and $U \cap A_2$ is open in A_2. Let U_1 be the interior of the domain of f_1. Then for each j with $0 \leqq j \leqq m-1$

$$f_1^{-1}(U_1)$$

is open in A_2 and contains A_1. Hence

$$A_1 \subset \bigcap_{j=0}^{m-1} f_1^{-j}(U_1)$$

where $\bigcap_{j=0}^{m-1} f_1^{-j}(U_1)$ is an open subset of the domain of f_1^m. Hence A_1 lies in the interior of the domain of f_1^m.

Let X_2 be a compact subset of the domain of f_1^m which contains $A_1 = X_3$ in its interior. Then the family $(X_0, X_1, X_2, X_3, f_1)$ satisfies the hypotheses of Theorem 10 so that f_1 has a fixed point in X_2. Since f_1 is a restriction of f, f has a fixed point in X. Q.E.D.

We now obtain some specializations of Theorem 11:

THEOREM 12. *Let X be one of the following spaces*:

(1) $X = E - \bigcup_\gamma Y_\gamma$, *where E is an infinite dimensional Banach space, $\{Y_\gamma\}$ a locally finite family of mutually disjoint open balls of E.*

(2) $X = S(E) - \bigcup_\gamma Y_\gamma$, *where* $S(E)$ *is the unit sphere in an infinite-dimensional Banach space* E, $\{Y_\gamma\}$ *a locally finite family of mutually disjoint open subsets of* $S(E)$ *such that for each* γ *there exists a homeomorphism* h_γ *of* Y_γ *on an open ball* $D(E_\gamma)$ *in an infinite-dimensional Banach space* E_γ *which can be extended to a homeomorphism of* $\bar{Y}_\gamma$ *on the closed ball mapping* $\dot{Y}_\gamma$ *homeomorphically on* $S(E_\gamma)$.

(3) $X = D(E) - \bigcup_\gamma Y_\gamma$, *where* $D(E)$ *is the open (or closed) unit ball in an infinite-dimensional Banach space* E *and* Y_γ *is a mutually disjoint locally finite family of open subsets of* $D(E)$ *of the same type as in* (2).

Let f *be a locally compact self-mapping of* X *such that for some* $m \geqq 1$, $f^m(X)$ *is relatively compact in* X.

Then f *has a fixed point in* X.

Proof of Theorem 12. It suffices by Theorem 11 to show that each of the types of spaces X of Theorem 12 can be imbedded as a retract of a convex subset of a Banach space which is either open or closed. We apply the theorem of Dugundji [9] that in an infinite-dimensional Banach space E, $S(E)$ is a retract of the closed unit ball $D(E)$. Hence E, $D(E)$ and $S(E)$ all satisfy the conditions of Theorem 11. Let X_0 be one of these and let $X = X_0 - \bigcup_\gamma Y_j$ where $\{Y_\gamma\}$ is a family satisfying the conditions of (2) of the hypothesis of Theorem 12. By Dugundji's theorem, there exists a retraction of $Y_\gamma \cup \dot{Y}_\gamma$ on $\dot{Y}_\gamma$ and hence a retraction of X_0 on X. Since a retract of a retract is a retract, our proof is complete. Q.E.D.

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